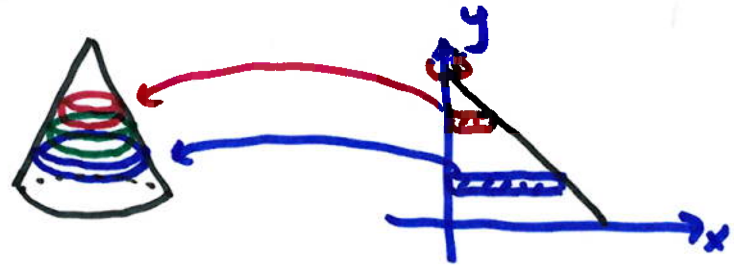
6.4 Volumes by Shells

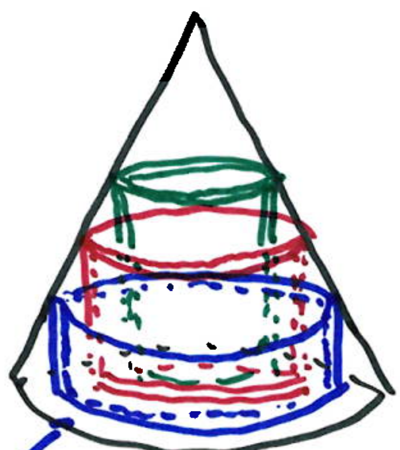
last time: method of disk/washer

stack volumes of thin disks



disk/washer: rectangles perpendicular to axis of revolution

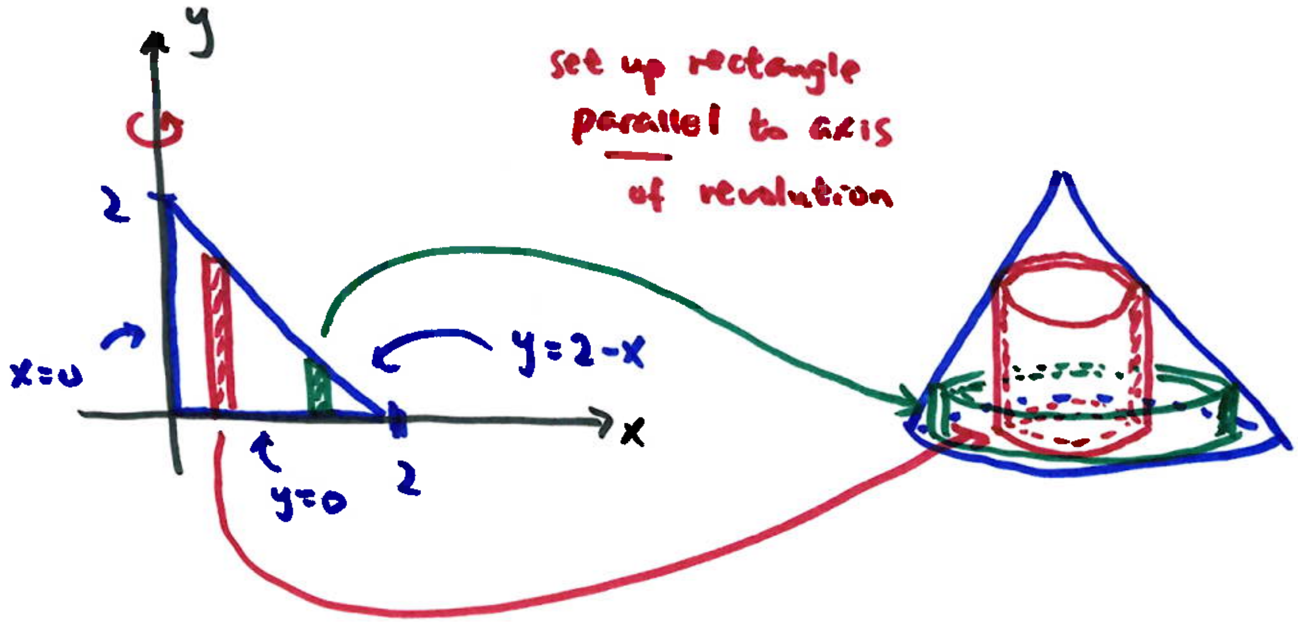
today: method of cylindrical shells



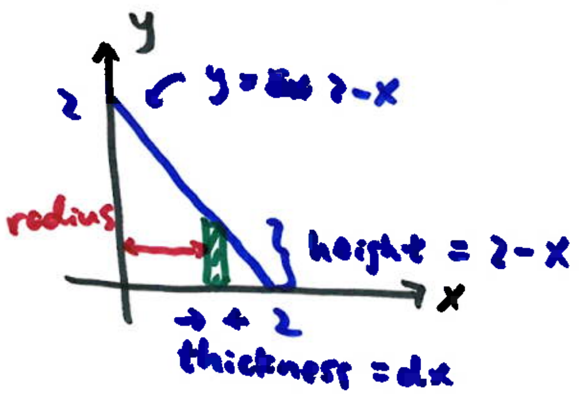
stacking infinitely-many shells to form, in this case, a cone

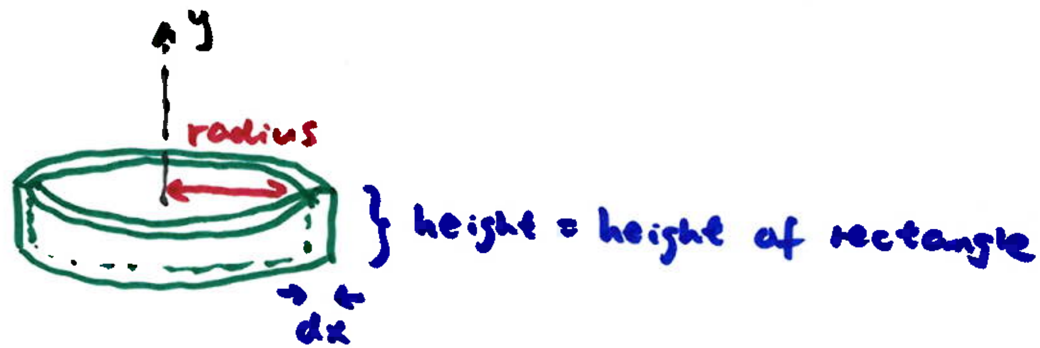
thin shell (think of onion layers)

example Use shells to find volume of solid obtained by revolving the region bounded by $y=2-x$, $y=0$, $x=0$, revolved around y-axis.



calculate volume of one shell, then integrate to accumulate all shells

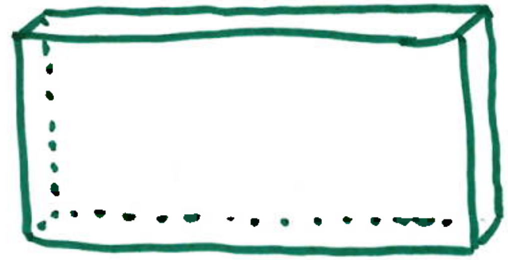




radius = how far rectangle is from axis of revolution

here, radius = x

volume of green shell: unwrap it



height = 2 - x

thickness = dx

length = circumference

= $2\pi \cdot$ radius

= $2\pi \cdot x$

volume = $(2\pi x)(2-x)(dx) = 2\pi x(2-x)\,dx$

this is one very thin rectangle / shell

we want to accumulate ALL such shells

start at $x=0$, end at $x=2$

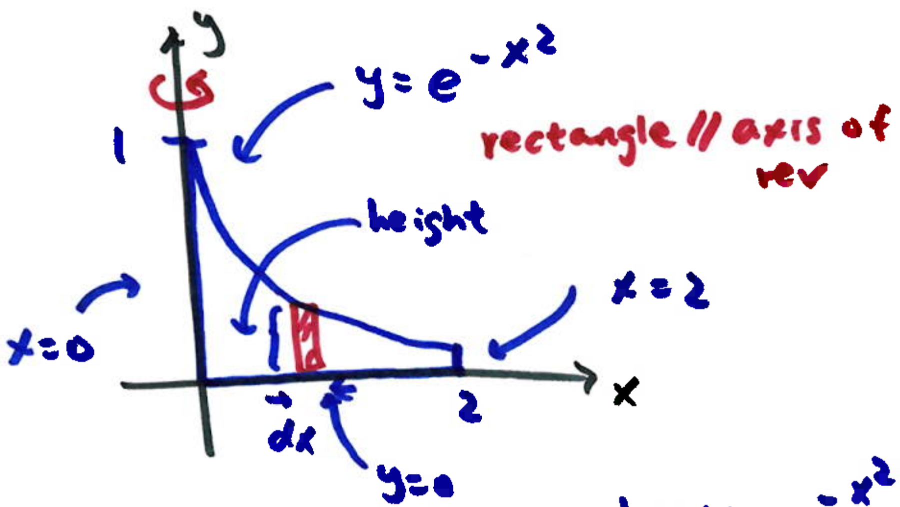
volume of the resulting cone

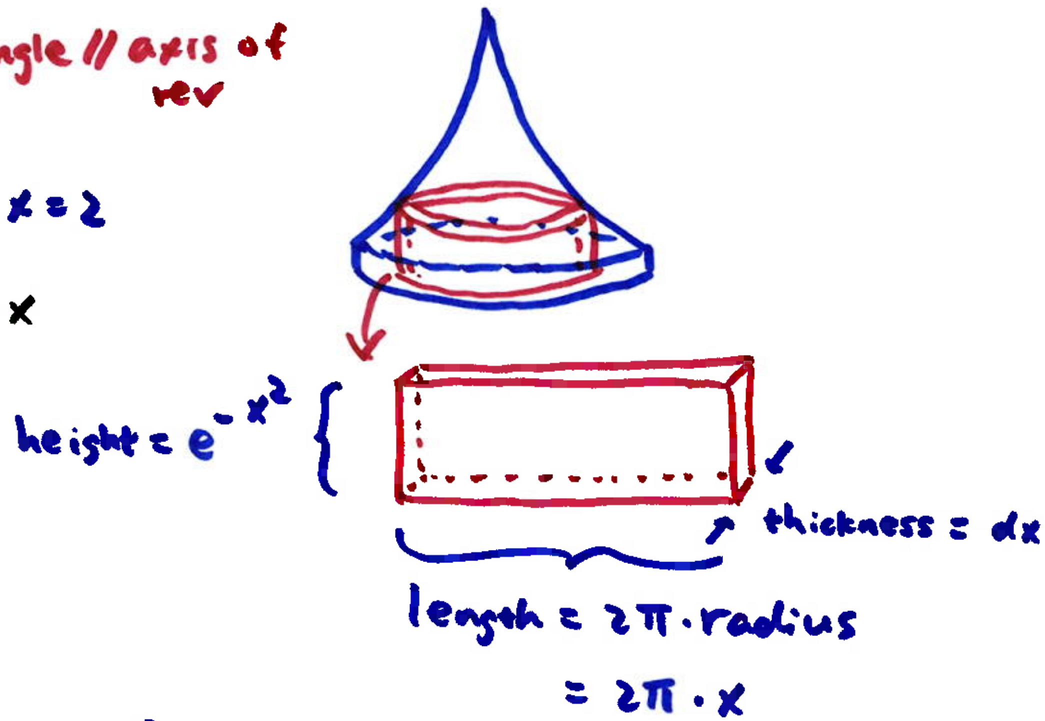
2 ← right end of region

$$= \int_0^2 \underbrace{2\pi x(2-x)\,dx}_{\text{one shell}} = \dots = \boxed{\frac{8\pi}{3}}$$

0 ↖ left end of region

Example Region bounded by $y = e^{-x^2}$, $x=0$, $y=0$, $x=2$ revolved around y-axis





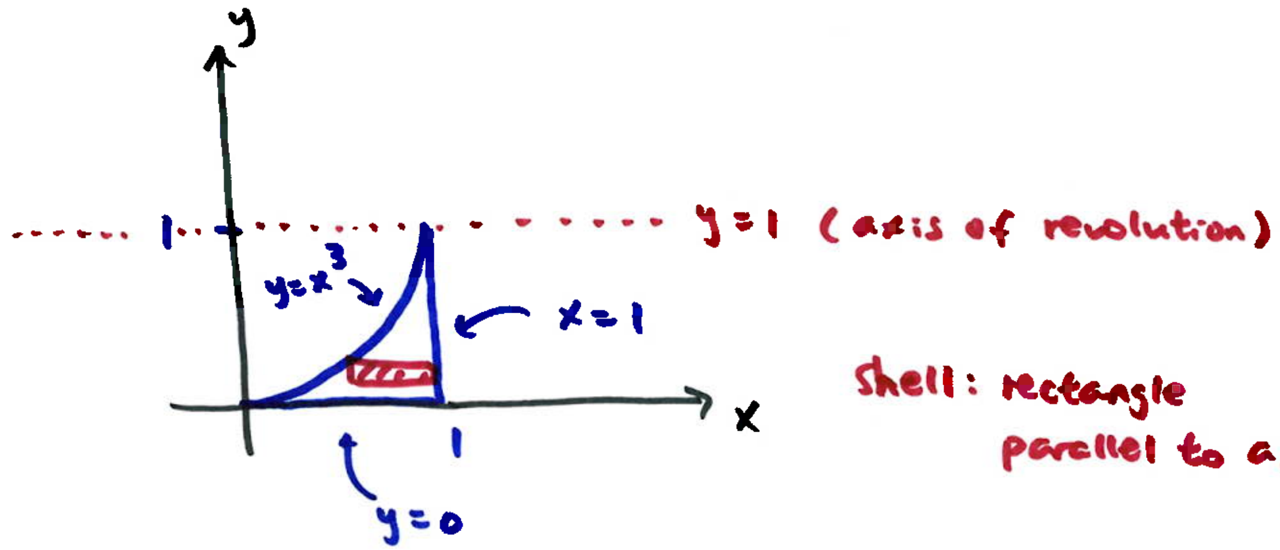
volume of one shell: $2\pi x e^{-x^2} dx$

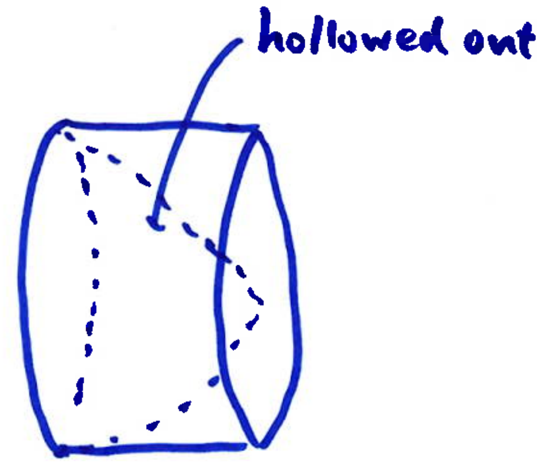
volume of object: sum all thin shell volumes by integration

$$\int_{0\ \leftarrow \text{left end of region}}^{2\ \leftarrow \text{right end of region}} \underbrace{2\pi x e^{-x^2} dx}_{\text{one shell}} = \text{by subs } \cancel{y = e^{-x^2}}\ u = -x^2 \text{ etc } = \cdots = \boxed{\pi\left(1 - e^{-4}\right)}$$

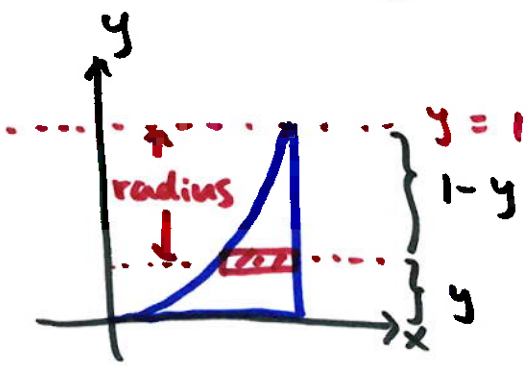
Example Region bounded by $y = x^3$, $x = 1$, $y = 0$

revolved around $y = 1$





shell: rectangle
parallel to axis of rev



radius = distance of rectangle from axis of rev.

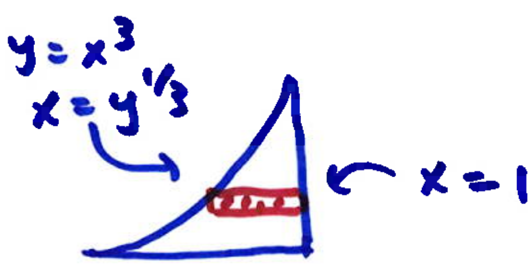
$= 1-y$

thickness $= dy$ ← integrate in terms of y
NO x in any part

"height" $= 1 - y^{1/3}$

right curve ($x=1$)

left curve ($y=x^3$ but turned into $x=y^{1/3}$)



volume of one shell = $2\pi\,(1-y)\,(1-y^{1/3})\,dy$

radius "height" thickness

accumulate ALL by integration from $y=0$ to $y=1$

(bottom of region) (top of region)

$$\int_0^1 2\pi(1-y)(1-y^{1/3})\,dy = \cdots = \boxed{\frac{5\pi}{14}}$$